


## SOLUTION OF THE DIFFUSION EQUATION: APPLICATIONS IN MODELING HEAT DIFFUSION IN SOIL

### SOLUÇÃO DA EQUAÇÃO DE DIFUSÃO: APLICAÇÕES NA MODELAGEM DA DIFUSÃO DE CALOR NO SOLO

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**Abstract:** This work presents an alternative approach for solving an initial and boundary value problem related to the one-dimensional non-homogeneous diffusion equation with constant coefficients, applied to soil heat conduction. Unlike the traditional Fourier variable separation method, the proposed approach is based on the fundamental solution (Green's function), the antisymmetric extension of the initial condition, and Duhamel's and superposition principles. The methodology involves analyzing the Cauchy problem for the heat equation in unbounded and semi-infinite domains, employing techniques of variable transformation and integration. Preliminary results indicate that this technique can be effectively applied in future studies on ground-air heat exchangers, aiming at thermal comfort in enclosed environments. This study represents an important initial step toward developing more efficient and accurate solutions to describe soil thermal behavior.

**Keywords:** Heat equation. Duhamel's principle. Green's function. Soil heat conduction.

#### Resumo:

Este trabalho apresenta uma abordagem alternativa para a solução de um problema de valores iniciais e de contorno relacionado à equação de difusão unidimensional não homogênea com coeficientes constantes, aplicada à condução de calor no solo. Ao contrário do método tradicional de separação de variáveis de Fourier, a abordagem proposta baseia-se na solução fundamental (função de Green), na extensão antissimétrica da condição inicial, e nos princípios de Duhamel e de superposição. A metodologia envolve a análise do problema de Cauchy para a equação do calor em domínios ilimitados e semi-infinitos, utilizando técnicas de transformação de variáveis e integração. Os resultados preliminares indicam que essa técnica pode ser aplicada efetivamente em estudos futuros sobre trocadores de calor solo-ar, visando conforto térmico em ambientes fechados. Este estudo é um passo inicial importante para o desenvolvimento de soluções mais eficientes e precisas que descrevem o comportamento do solo.

**Palavras-chave:** Equação do calor. Princípio de Duhamel. Função de Green. Condução de calor no solo.

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# 1. INTRODUCTION

The analysis of diffusive phenomena is essential for understanding various natural and industrial processes. This study proposes an alternative approach to obtaining a closed-form expression for the exact solution of an initial and boundary value problem for the one-dimensional non-homogeneous diffusion equation with constant coefficients in a semi-infinite medium. Unlike the Fourier variable separation method (BRUM, 2013), the adopted approach is based on the fundamental solution, the antisymmetric extension of the initial condition, and the principles of Duhamel and superposition (LOGAN, 2015).

Partial differential equations often arise from conservation laws, which state that certain physical properties of an isolated system remain invariant over time. Diffusion, as a transport phenomenon, describes how a conserved quantity, such as energy or mass, is distributed in space over time due to concentration gradients (LOGAN, 2015).

The relevance of this study lies in the practical application of the obtained solutions to real problems, such as in spatial ecology and evolutionary biology (LAM; LOU, 2022), building physics (MENDES et al., 2019), heat conduction (HAM; ÖZISIK, 2012), atmospheric phenomena (HANNA; BRIGGS; HOSKER, 1982), including meteorological processes (STULL, 1988), pollutant dispersion (ARYA, 1998; MING et al., 2017), and image restoration and interpolation (BARBU, 2019). The proposed methodology broadens the theoretical understanding of diffusive phenomena and provides tools for applied engineering, improving the efficiency of thermal systems in various contexts.

Furthermore, as described by Feynman in his renowned *Lectures on Physics* (FEYNMAN; LEIGHTON; SANDS, 2013), diffusive phenomena play a crucial role across multiple scientific domains. Diffusion is fundamental in explaining processes such as heat transfer in solid materials, electrical conduction in semiconductors, particle propagation in heterogeneous media, and dielectric polarization. Feynman also highlights the importance of diffusion in chemical

processes, such as the mixing of substances in solutions. These examples illustrate the versatility of diffusive phenomena and the relevance of understanding them to advance multiple areas of scientific and technological knowledge.

This work investigates the solution of the heat diffusion equation in soil with a null source term and non-homogeneous boundary conditions, aiming to compare the efficiency of different solution methods. The methods discussed, particularly the approach based on the fundamental solution, proved to be effective and accurate. The computational implementation of the solutions, using the Python software and scientific libraries such as NumPy, SciPy, and Matplotlib, enabled a detailed analysis and visualization of diffusive phenomena. This comparative analysis highlights the effectiveness of the proposed approach relative to other solution methods for diffusion phenomena.

Section 2 discusses the analysis of partial differential equations in different domains. Both theoretical and computational approaches are employed to solve classical and non-classical diffusion problems. Section 3 presents the results obtained, focusing on soil heat diffusion and the computational modeling of the studied phenomena, simulating temperature distribution under different conditions. Finally, Section 4 discusses the results, emphasizing the theoretical and practical implications of the presented solutions.

## **2 DEVELOPMENT**

### **2.1 DIFFUSION EQUATIONS IN UNBOUNDED DOMAINS**

The choice of the spatial domain is crucial in problem solving, as it can be either bounded or unbounded. The preference for a particular type of domain depends on the characteristics of the problem. According to Logan (2015), although problems in bounded domains are common, the unbounded domain approach offers advantages such as simplifying the problem formulation and reducing the complexity of boundary conditions. Understanding the categories of partial

differential equations and considering the spatial domain are fundamental for an effective solution. This initial approach can simplify the analysis and provide a solid foundation for further investigations.

Consider the following initial value problem (IVP) for the homogeneous diffusion equation with a non-homogeneous initial condition, expressed as follows:

$$\begin{aligned} u_t - ku_{xx} &= 0, & (x, t) \in \mathbb{R} \times \mathbb{R}_+^*, \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}, \end{aligned} \quad (1)$$

where  $u$  represents the temperature and  $k$  is the thermal diffusivity constant. This problem models heat propagation in an infinite bar with an initial temperature distribution  $u_0(x)$ . In order to solve (1), the following problem is considered:

$$\begin{aligned} w_t - kw_{xx} &= 0, & (x, t) \in \mathbb{R} \times \mathbb{R}_+^*, \\ w(x, 0) &= w_0 H(x), & x \in \mathbb{R}, \end{aligned} \quad (2)$$

where  $w_0$  is a constant and  $H(x)$  is the Heaviside unit step function.

The solution to the auxiliary problem (2) is given by:

$$w(x, t) = \frac{w_0}{2} \left( 1 + \operatorname{erf} \left( \frac{x}{\sqrt{4kt}} \right) \right), \quad (3)$$

where the Gaussian error function is defined as,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr. \quad (4)$$

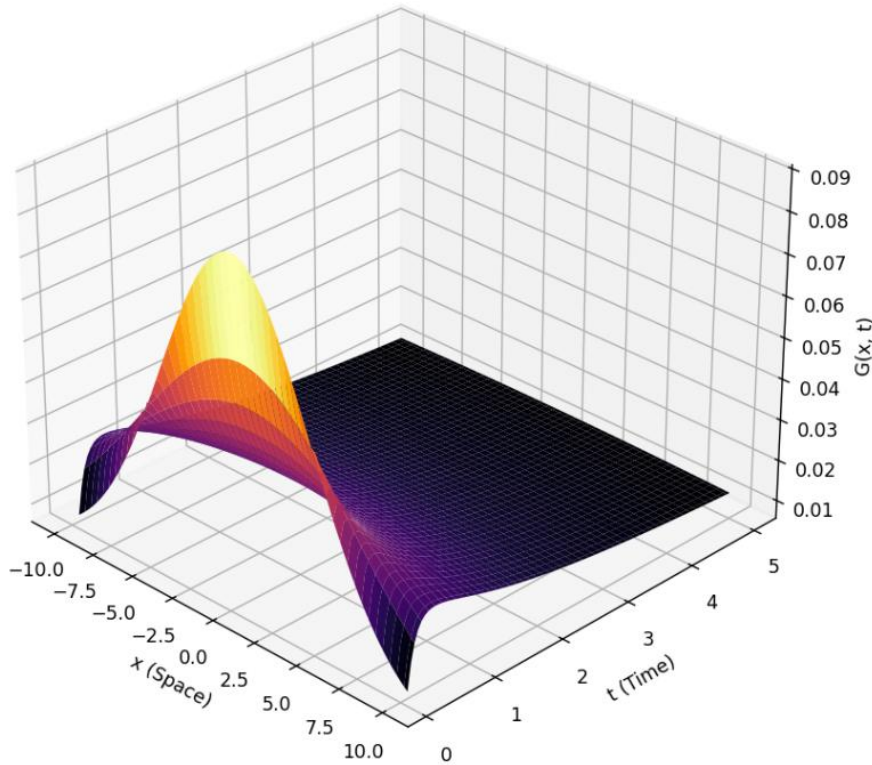
Also, note that if  $w$  is a solution to the heat equation, then  $w_x$  is also a solution. Therefore, for  $w_0 = 1$ , we have

$$w(x, t) = \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{x}{\sqrt{4kt}} \right) \right) \Rightarrow w_x = \frac{1}{\sqrt{4\pi kt}} \exp \left( -\frac{x^2}{4kt} \right) = G(x, t). \quad (5)$$

The function  $G(x, t)$  is called the heat kernel, Green's function for the heat equation, or fundamental solution of the heat equation. It represents the temperature distribution that results from a point heat source located at  $x = 0$  and that initially transfers a unit of heat  $G(x, 0) = w_0 = 1$  (see figure 1). The function

$G(x - y, t)$  represents such a temperature distribution if the heat source is located at  $x = y$ .

**Figure 1** – Fundamental solution  $G(x, t)$  for  $k = 100$ .



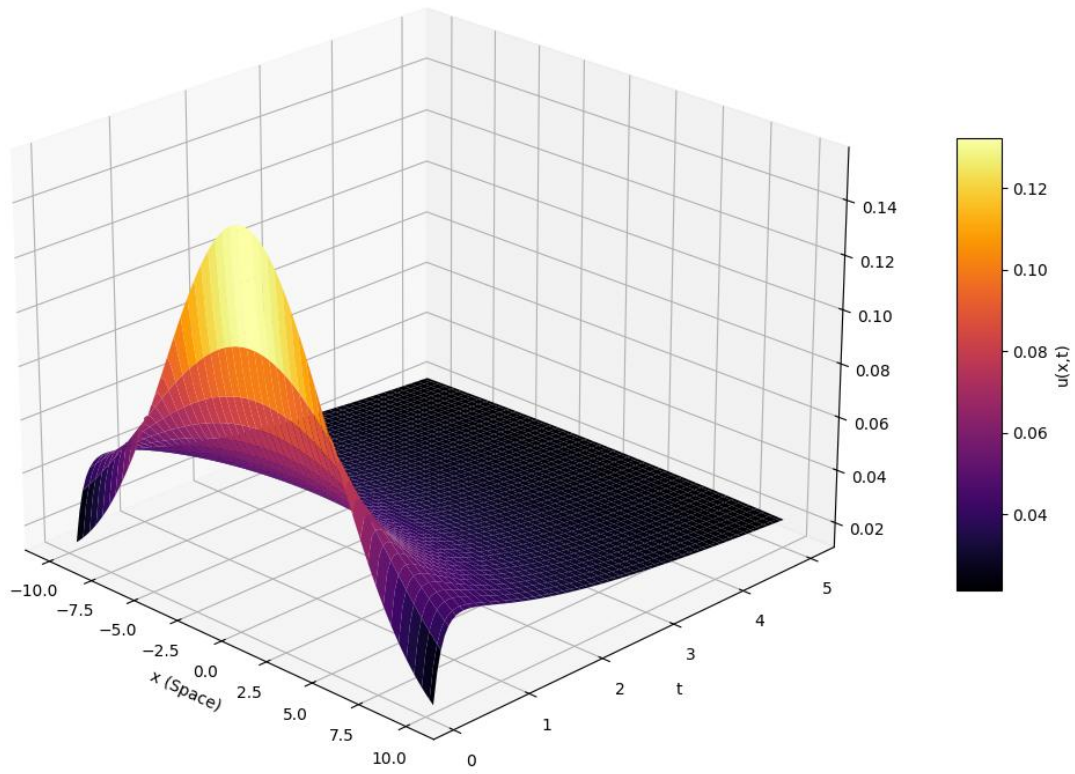
**Source:** From the authors.

On the other hand, if  $u_0(y)$  is the amount of heat transferred at  $x = y$  then  $w_0 = u_0(y)$  and  $u_0(y)G(x - y, t)$  the corresponding solution of the heat equation. If  $u_0(y)$  is continuous and bounded, then superposing the effects of all punctual sources  $u_0(y)$ ,  $y \in \mathbb{R}$ , produces the solution of the original problem as

$$u(x, t) = \int_{-\infty}^{+\infty} u_0(y)G(x - y, t)dy = \int_{-\infty}^{+\infty} \frac{u_0(y)}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} dy. \quad (6)$$

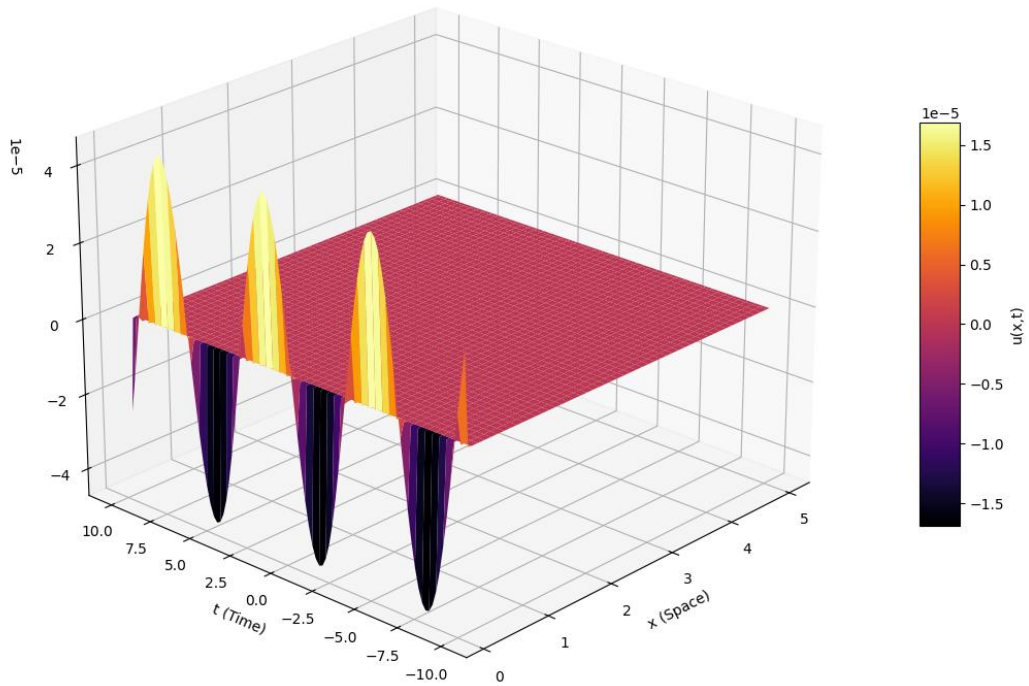
Figures 2, 3 and 4 show the behavior of the solution (6) for different values of diffusivity  $k$  and three different initial conditions: Gaussian, sinusoidal and arctangent, respectively.

**Figure 2** – Solution  $u(x, t)$  of the IVP with  $k = 100$  and  $u_0(x) = e^{-x^2}$ .



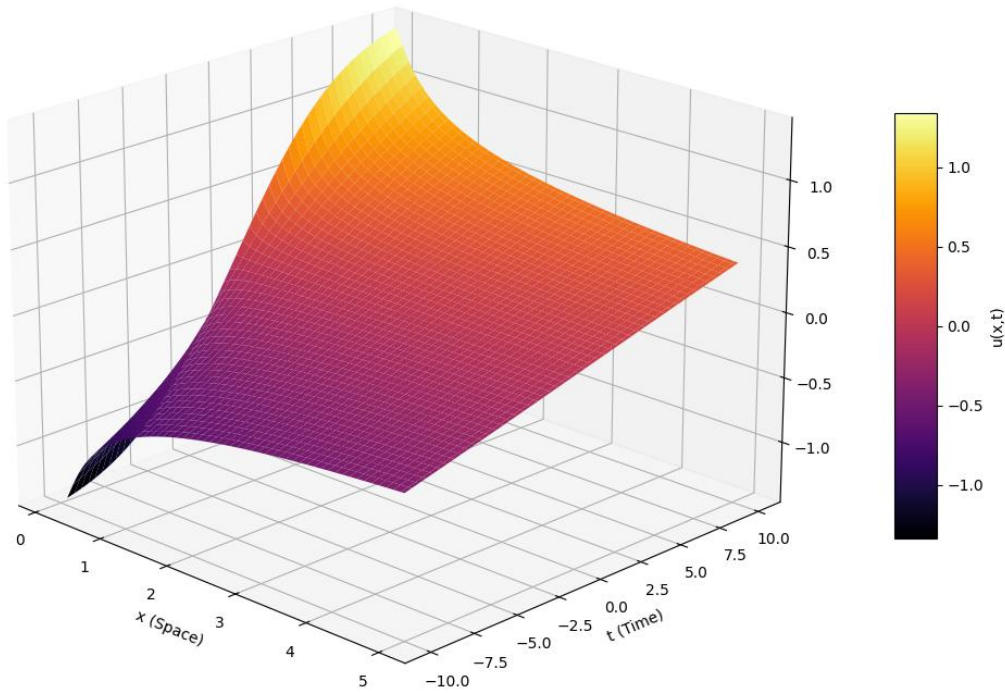
**Source:** From the authors.

**Figure 3** – Solution  $u(x, t)$  of the IVP with  $k = 100$  and  $u_0(x) = \sin(x)$ .



**Source:** From the authors.

**Figure 4** – Solution  $u(x, t)$  of the IVP with  $k = 100$  and  $u_0(x) = \arctan(x)$ .



**Source:** From the authors.

## 2.2 DIFFUSION EQUATIONS IN SEMI-INFINITE DOMAINS

Consider the following initial- and boundary-value problem for the non-homogeneous diffusion equation.

$$\begin{aligned}
 u_t - ku_{xx} &= 0, & (x, t) &\in \mathbb{R}_+^* \times \mathbb{R}_+^*, \\
 u(0, t) &= 0, & t &\in \mathbb{R}_+, \\
 u(x, 0) &= u_0(x), & x &\in \mathbb{R}_+,
 \end{aligned} \tag{7}$$

with  $u_0(0) = 0$  for compatibility with the boundary condition. According to Logan (2015), the solution of problem (7) is obtained from the solution (6) of the Cauchy problem (1). Specifically, consider the auxiliary variable  $v(x, t): \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ , where  $v = u$  for  $x > 0$  being  $u$  the solution of problem (7). Let  $v_0$  be the antisymmetric extension to all  $\mathbb{R}$  of the initial condition  $u^0$ . Thus, the solution  $v(x, t)$  of the diffusion equation  $v_t - kv_{xx} = 0$ ,  $x \in \mathbb{R}$ ,  $t > 0$  which meets the initial condition  $v(x, 0) = v_0(x)$  is given by:

$$\begin{aligned}
v(x, t) &= \int_{-\infty}^{+\infty} v_0(y)G(x - y, t)dy = - \int_{-\infty}^0 u_0(-y)G(x - y, t)dy + \int_0^{+\infty} u_0(y)G(x - y, t)dy \\
&= \int_0^{+\infty} u_0(y)[G(x - y, t) - G(x + y, t)]dy, \quad x \in \mathbb{R}. \tag{8}
\end{aligned}$$

Therefore, the solution  $u(x, t)$  of the original problem (7), obtained by restricting the solution  $v(x, t)$  to the region  $\mathbb{R}_+$ , is given by:

$$u(x, t) = \int_0^{+\infty} u_0(y)[G(x - y, t) - G(x + y, t)]dy, \quad x \in \mathbb{R}. \tag{9}$$

### 2.2.1 Duhamel's Principle for Problems with Non-Zero Source Term

Consider the Cauchy problem for a partial differential equation with a nonzero source term and a homogeneous initial condition:

$$u_t - ku_{xx} = f(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = 0, \quad x \in \mathbb{R} \tag{10}$$

From the Duhamel's principle, the solution is:

$$u(x, t) = \int_0^t u^{(\tau)}(y, \tau)G(x, t - \tau)d\tau \tag{11}$$

where  $u^{(\tau)}(x, t)$  is the solution of the Cauchy problem for the diffusion equation with zero source term and inhomogeneous initial condition given by the source term of the original problem:

$$\begin{aligned}
u_t^{(\tau)} - ku_{xx}^{(\tau)} &= 0, \quad x \in \mathbb{R}, \\
u^{(\tau)}(x, 0) &= f(x, \tau).
\end{aligned} \tag{12}$$

whose solution is:

$$u^{(\tau)}(x, t) = \int_{-\infty}^{+\infty} f(y, \tau)G(x - y, t)dy \tag{13}$$

so the solution to the original problem is:

$$u(x, t) = \int_0^t \int_{-\infty}^{+\infty} f(y, \tau) G(x - y, t - \tau) dy d\tau \quad (14)$$

### 2.2.2 Solution of the Diffusion Equation with Non-Zero Source Term and Non-Homogeneous Initial Condition

Consider the Cauchy problem for the diffusion equation with non-zero source and inhomogeneous initial condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x, t), \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \quad (15)$$

Given the linearity of the diffusion equation, its solution can be obtained via the superposition principle  $u = u^{(1)} + u^{(2)}$  and by solving auxiliary Cauchy problems for  $u^{(1)}$  and  $u^{(2)}$ , defined as follows:

$$\begin{aligned} u_t^{(1)} - ku_{xx}^{(1)} &= f(x, t), \quad x \in \mathbb{R}, \quad t > 0, \\ u^{(1)}(x, 0) &= 0, \quad x \in \mathbb{R}. \end{aligned} \quad (16)$$

and

$$\begin{aligned} u_t^{(2)} - ku_{xx}^{(2)} &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u^{(2)}(x, 0) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \quad (17)$$

The solution to the original problem can then be expressed as:

$$u(x, t) = \int_0^t \int_{-\infty}^{+\infty} f(y, \tau) G(x - y, t - \tau) dy d\tau + \int_{-\infty}^{+\infty} u_0(y) G(x - y, t) dy \quad (18)$$

where (14) and (6) are the solutions  $u^{(1)}$  and  $u^{(2)}$  of (16) and (17), respectively.

Next, we consider a new problem, now without a source term, but with non-homogeneous conditions, formulated as follows:

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(0, t) &= u^0(t), \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x > 0. \end{aligned} \quad (19)$$

In order to homogenize the boundary condition, we introduce a new variable  $v(x, t) = u(x, t) - u^0(t)$ , so obtaining the following Cauchy problem:

$$\begin{aligned} v_t - kv_{xx} &= -u_t^0, & x > 0, \quad t > 0, \\ v(0, t) &= 0, & t > 0, \\ u(x, 0) = v_0(x) &= u_0(x) - u^0(0), & x > 0. \end{aligned} \quad (20)$$

Given the linearity of the diffusion equation, the solution will be the superposition of the solutions of the auxiliary problems, therefore:

$$\begin{aligned} v_t^{(1)} - kv_{xx}^{(1)} &= -u_t^0, & x > 0, \quad t > 0, \\ v^{(1)}(0, t) &= 0, & t > 0, \\ v^{(1)}(x, 0) &= 0, & x > 0. \end{aligned} \quad (21)$$

and

$$\begin{aligned} v_t^{(2)} - kv_{xx}^{(2)} &= 0, & x > 0, \quad t > 0, \\ v^{(2)}(0, t) &= 0, & t > 0, \\ v^{(2)}(x, 0) &= v_0(x), & x > 0. \end{aligned} \quad (22)$$

The solution to the first auxiliary problem is given by Duhamel's principle as

$$v^{(1)}(x, t) = \int_0^{+\infty} \int_0^t u_t^0(\tau) [G(x - y, t - \tau) - G(x + y, t - \tau)] dy d\tau. \quad (23)$$

The solution to the second auxiliary Cauchy problem will be:

$$v^{(2)}(x, t) = \int_0^{+\infty} v_0(y) [G(x - y, t) - G(x + y, t)] dy. \quad (24)$$

And the solution to the original problem is:

$$u(x, t) = \int_0^{+\infty} \int_0^t u_t^0(t)[G(x - y, t - \tau) - G(x + y, t - \tau)] dy d\tau + \int_0^{+\infty} u^0(y)[G(x - y, t) - G(x + y, t)] dy. \quad (25)$$

### 3 RESULTS

#### 3.1 HEAT DIFFUSION IN THE SOIL

The phenomenon of heat diffusion in the soil results from the IBVP (19), which, in turn, varies only with respect to time and depth  $x = z$  when certain assumptions are made, namely that the soil is in natural conditions, soil stratification increases as depth increases, and that there is no internal heat generation (i.e.,  $f(z, t) = 0$ ).

According to Brum (2013) at the surface where  $z = 0$ , the boundary condition for the heat diffusion equation is given by

$$u(0, t) = T_0 + \theta_0 \sin(\omega t), \quad t > 0. \quad (26)$$

We use the initial condition obtained by putting  $t = 0$  in the solution obtained by Brum (2013) via separation of variables given by

$$\bar{u}(z, t) = T_0 + \theta_0 \exp\left(-\sqrt{\frac{\omega}{2k}}z\right) \left[ \sin\left(\omega t - \sqrt{\frac{\omega}{2k}}z\right) \right], \quad t > 0, \quad z > 0, \quad (27)$$

where  $T_0$  is the mean temperature,  $\theta_0$  is the amplitude of the variation, and the period  $\omega$  varies either the daily or annual modulus.

Therefore, the initial value and boundary problem based on the established conditions will be:

$$\begin{aligned} u_t - ku_{zz} &= 0, & z > 0, \quad t > 0, \\ u(0, t) &= u^0(t) = T_0 + \theta_0 \sin(\omega t), & t > 0, \\ u(z, 0) &= T_0 - \theta_0 \exp\left(-\sqrt{\frac{\omega}{2k}}z\right) \left[ \sin\left(\sqrt{\frac{\omega}{2k}}z\right) \right], & z > 0. \end{aligned} \quad (28)$$

Defining  $v(z, t) = u(z, t) - u^0(t)$  to homogenize the boundary condition, the following IBVP with non-zero source is obtained:

$$\begin{aligned} v_t - kv_{zz} &= -\omega\theta_0 \cos(\omega t), & t > 0, z > 0, \\ v(0, t) &= 0, & t > 0, \\ v(z, 0) &= v_0(z) = u_0(z) - T_0, & z > 0. \end{aligned} \quad (29)$$

Given the linearity of the heat diffusion equation and the superposition principle, we have  $v(z, t) = v^{(1)} + v^{(2)}$  where  $v^{(1)}$  and  $v^{(2)}$  are solutions of the auxiliary problems, where

$$\begin{aligned} v_t^{(1)} - kv_{zz}^{(1)} &= -\omega\theta_0 \cos(\omega t), & t > 0, z > 0, \\ v^{(1)}(0, t) &= 0, & t > 0, \\ v^{(1)}(z, 0) &= 0, & z > 0, \end{aligned} \quad (30)$$

and

$$\begin{aligned} v_t^{(2)} - kv_{zz}^{(2)} &= 0, & t > 0, z > 0, \\ v^{(2)}(0, t) &= 0, & t > 0, \\ v^{(2)}(z, 0) &= v_0(z), & z > 0, \end{aligned} \quad (31)$$

which are of the same types as (21) and (22), respectively. Then, the solution to the original problem (28) is:

$$\begin{aligned} u(x, t) &= -\omega\theta_0 \int_0^{+\infty} \int_0^t \cos(\omega\tau) [G(x-y, t-\tau) - G(x+y, t-\tau)] dy d\tau \\ &- \int_0^{+\infty} \theta_0 e^{-\sqrt{\frac{\omega}{2k}}y} \sin\left(\sqrt{\frac{\omega}{2k}}y\right) [G(x-y, t) - G(x+y, t)] dy + T_0 + \theta_0 \sin(\omega t). \end{aligned} \quad (32)$$

### 3.2 COMPUTATIONAL MODELING

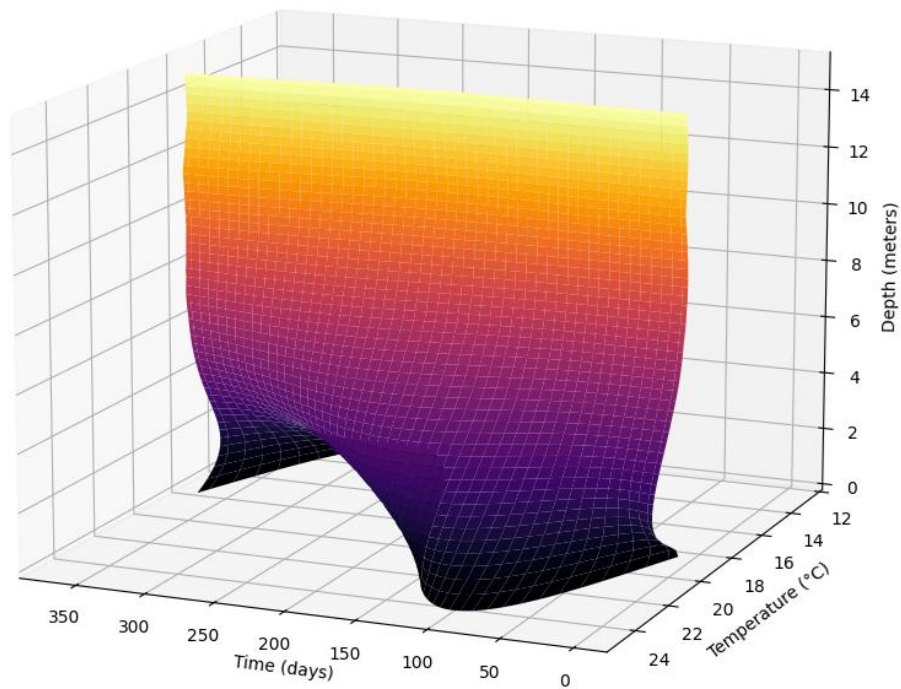
Using solution (32), the analytical solution was implemented to provide a detailed representation of heat propagation over time and depth, comparing it with alternative approaches, such as the method used by Brum (2013).

To implement the analytical solution, the Python programming language was used, employing the NumPy, SciPy, and Matplotlib libraries for numerical calculations, integration, and visualization, respectively. In order to facilitate

reproduction of results, the codes are provided in the Appendix. Using the developed algorithms, it was possible to perform three types of analyses.

In figure 5, it is possible to simultaneously observe the variation of temperature in relation to time and depth. This solution can also be obtained in  $\mathbb{R}_+^3$ .

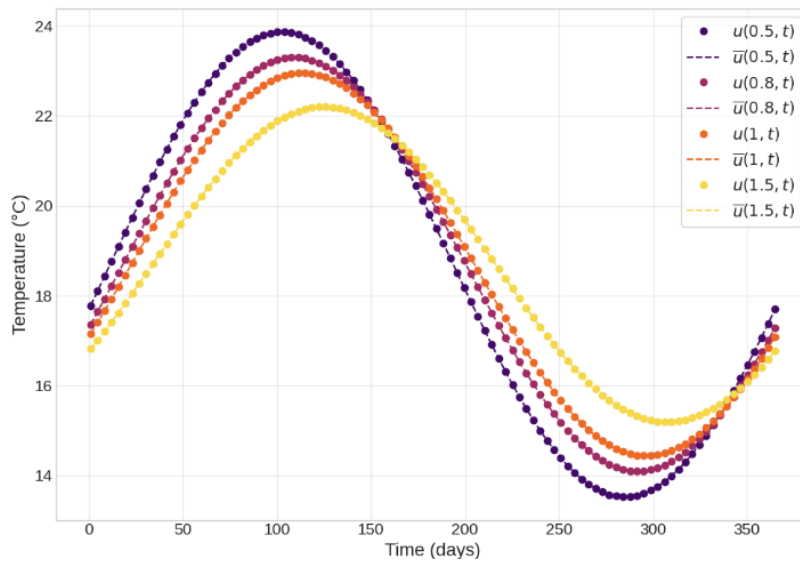
**Figure 5** – Temperature variation  $u(z, t)$ .



**Source:** From the authors.

Figure 6 presents a comparison between equations (27) and (32) to analyze temperature variation over time.  $\theta_0 = 6.28$ , the period  $\omega = 2\pi/365$  and the diffusivity constant  $k = 0.057 \text{ mm}^2/\text{s}$  were considered. It was observed that the temperature variation is coincident at all points, with the same behavior being observed for the temperature variation in relation to depth. The solution  $u(z, t)$  is represented by circular markers, while Brum (2013) solution is represented by continuous curves.

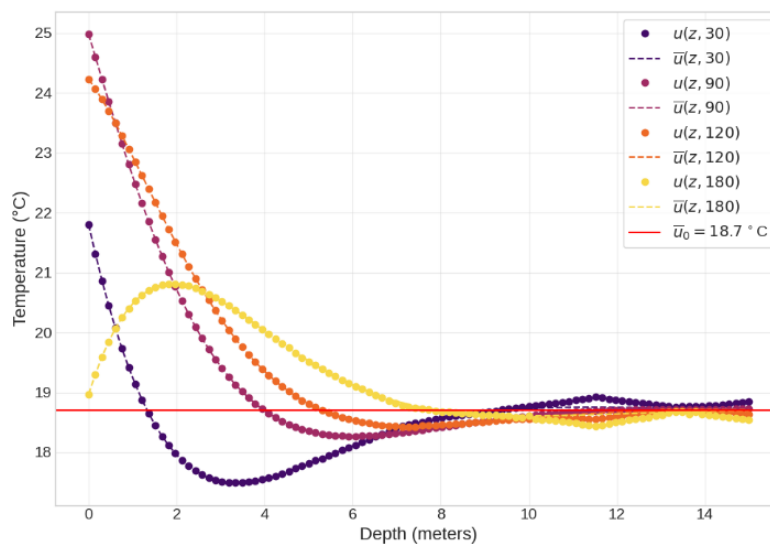
**Figure 6** – Comparison between solutions  $u(z, t)$  and  $\bar{u}(z, t)$ .



**Source:** From the authors.

In figure 7, it can be seen that, as depth increases, the temperature tends towards an average value of 18,7 °C.

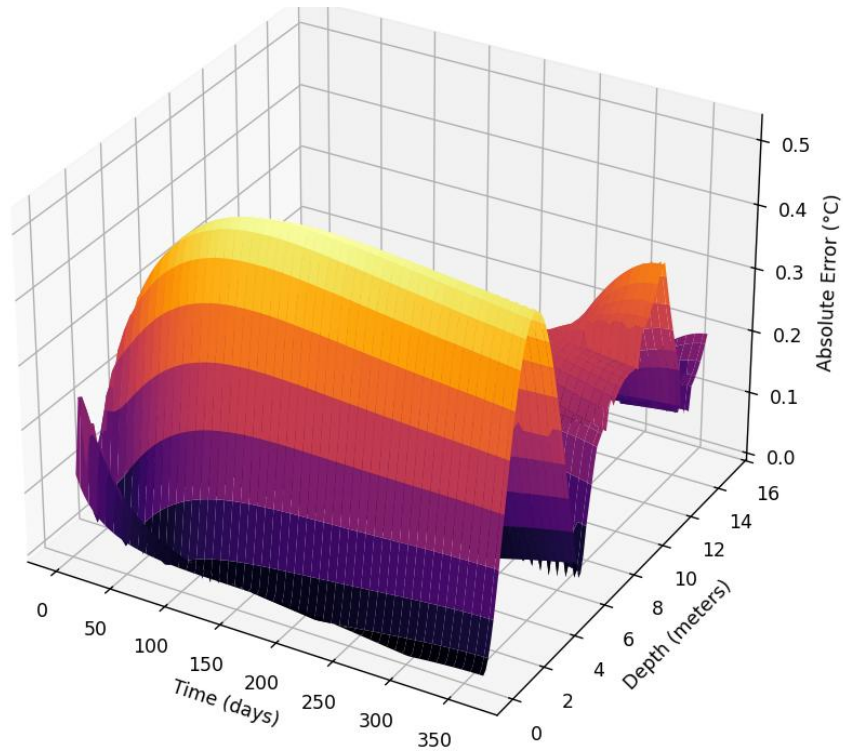
**Figure 7** – Comparison between solutions  $u(z, t)$  and  $\bar{u}(z, t)$ .



**Source:** From the authors.

Visualizing the variation  $e(z, t) = |u(z, t) - \bar{u}(z, t)|$  indicates that  $0 \leq e(z, t) \leq 0.1975$  °C, as illustrated in figure 8.

**Figure 8** – Error variation  $e(z, t)$ .



**Source:** From the authors.

## 4 FINAL CONSIDERATIONS

The study demonstrates the effectiveness of the Duhamel's principle approach in modeling heat diffusion in soil. The analytical solution obtained offers a detailed representation of heat propagation, allowing a deeper understanding of the phenomenon studied. Furthermore, the comparison with the Fourier method, as used by Brum (2013) highlights the advantages and disadvantages of each approach in solving heat transfer problems.

While both methods have been effective in modeling heat diffusion in soil, the choice between them depends on the specific characteristics of the problem and the available computational resources. The Duhamel method may be more suitable in situations where boundary conditions and medium properties vary over time, while the Fourier method may be preferable in problems with symmetry and well-defined boundary conditions.

In summary, the study contributes to the advancement of knowledge in the field of soil heat transfer, providing a solid foundation for future research and practical applications in various fields of engineering and environmental sciences.

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## APPENDIX

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from scipy.integrate import quad, dblquad
```

```
k = 0.057 # Soil diffusivity constant
T = 365 # Time in days
omega = 2 * np.pi / T # Period
```

```

theta0 = 6.28 # Amplitude
u0 = 18.7 # Mean soil temperature
# Defining the function to be integrated
def v1_zt(y, tau, t, z):
    return np.cos(omega * tau + theta0) * (
        (1 / np.sqrt(4 * np.pi * k * (t - tau))) * np.exp(-((z - y)**2) / (4 * k * (t - tau))) -
        (1 / np.sqrt(4 * np.pi * k * (t - tau))) * np.exp(-((z + y)**2) / (4 * k * (t - tau))) )
# Defining the function for the second integral
def v2_zt(y, t):
    return (1 / np.sqrt(4 * np.pi * k * t)) * (np.exp(-((z - y)**2) / (4 * k * t)) - np.exp(-
((z + y)**2) / (4 * k * t)))
# Defining the temperature function with the additional contribution of the
second integral
def temperatura_u(t, z):
    # Calculating the first integral
    integral1, _ = dblquad(v1_zt, 0.1, t, lambda tau: 0.1, lambda tau: np.inf, args=(t,
z)) # type: ignore
    # Calculating the second integral
    integral2, _ = quad(v2_zt, 0.1, np.inf, args=(t,))
    # Calculating the total temperature
    temperatura_total = -omega * theta0 * integral1 + (theta0 * np.exp(-
np.sqrt(omega/(2*k))*z) * np.sin(np.sqrt(omega/(2*k))*z))*integral2 + u0 + theta0
* np.sin(omega * t + theta0)

    return temperatura_total

# Defining time and depth values
t_values = np.linspace(0.1, 365, 100) # Time values from 0 to 365 days
z_values = np.linspace(0.1, 15, 100) # Depth values from 0 to 15 meters
# Calculating temperature values for each combination of time and depth
temperatura_values = np.zeros((len(z_values), len(t_values)))
for i, z in enumerate(z_values):
    for j, t in enumerate(t_values):
        temperatura_values[i, j] = temperatura_u(t, z)
# Plotting the 3D surface
T, Z = np.meshgrid(t_values, z_values)
fig = plt.figure(figsize=(10, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(T, temperatura_values, Z, cmap='inferno') # type: ignore
ax.set_xlabel('Time (days)')
ax.set_ylabel('Temperature (°C)')
ax.set_zlabel('Depth (meters)') # type: ignore
plt.show()

```